

Finite Difference Approximations

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Recall that the Taylor Series expansion of a function f about the point x is:

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \dots \\ f(x-h) &= f(x) - f'(x)h + \frac{f''(x)}{2!}h^2 + \dots \end{aligned} \quad (\text{replacing } h \text{ with } -h)$$

This can be extended to partial derivatives as well. In particular, consider the function $u(x, y, t)$.

$$u(x+h, y, t) = u(x, y) + u_x(x, y, t)h + u_{xx}(x, y, t)\frac{h^2}{2!} + \dots$$

If we truncate everything but the first two terms of the Taylor expansion, we obtain an approximation called the *first order Taylor series approximation*.

$$u(x+h, y, t) \approx u(x, y, t) + u_x(x, y, t)h \quad (1)$$

Solving for u_x gives us:

$$u_x(x, y, t) \approx \frac{u(x+h, y, t) - u(x, y, t)}{h} \quad (\text{Forward difference approx.})$$

We can also approximate u_{xx} by keeping another term of the expansion, i.e. by doing a second order Taylor series approximation.

$$\begin{aligned} u(x+h, y, t) &\approx u(x, y) + u_x(x, y, t)h + u_{xx}(x, y, t)\frac{h^2}{2!} && (\text{Keep an extra term}) \\ u_{xx}(x, y, t) &\approx \frac{2}{h^2}[u(x+h, y, t) - u(x, y) - u_x(x, y, t)h] && (\text{Solve for } u_{xx}) \end{aligned}$$

If we repeat the same procedure, replacing h with $-h$, we get the *backward difference approximation*. For example:

$$\begin{aligned} u_x(x, y, t) &\approx \frac{u(x, y, t) - u(x-h, y, t)}{h} && (\text{Solving for } u_x) \\ u_{xx}(x, y, t) &\approx \frac{2}{h^2}[u(x-h, y, t) - u(x, y) + u_x(x, y, t)h] && (\text{Solving for } u_{xx}) \end{aligned}$$

Subtracting the forward difference from the backward difference yields the *central difference approximation*. The central difference approximations for u_t , u_x , etc then are:

$$\begin{aligned} u_x(x, y, t) &\approx \frac{1}{2h}(u(x+h, y, t) - u(x-h, y, t)) \\ u_{xx}(x, y, t) &\approx \frac{1}{h^2}[u(x+h, y, t) - 2u(x, y) + u(x-h, y, t)] \\ u_y(x, y, t) &\approx \frac{1}{2h}(u(x, y+k, t) - u(x, y-k, t)) \\ u_{yy}(x, y, t) &\approx \frac{1}{k^2}[u(x, y+k, t) - 2u(x, y) + u(x, y-k, t)] \\ u_t(x, y, t) &\approx \frac{1}{r}(u(x, y, t+r) - u(x, y, t)) && (\text{Forward difference}) \end{aligned}$$